MAT102 Intro to Math Proofs

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Overview

AM-GM Inequality

Motivation

An investor has annual return of 5%, 10%, 20%, -50%, and 20% for five consecutive years. Using the arithmetic mean, the investor's average annual return rate is Does this number capture what happened in real life? Let's assume the investor started with \$1000.

Year	Starting Equity	Return %	Return \$	Closing equity
1	\$1 000	5%	\$50	\$
2	\$	10%	\$	\$
3	\$	20%	\$	\$
4	\$	-50%	-\$	\$
5	\$	20%	\$	\$

The actual 5 year return on the account is
The geometric mean is used to tackle continuous data series which the arithmetic mean is unable to accurately reflect.

Assume R_n =growth rate for year n, we define the geometric mean for investments to be

$$\left(\prod_{n=1}^{N}(1+R_n)\right)^{\frac{1}{N}}-1.$$

In the previous example, if we assume the average rate of return to be R_{ave}, then we should have

$$1000\times(1+R_{ave})^5 = 1000\times(1+5\%)(1+10\%)(1+20\%)(1-50\%)(1+20\%)$$

which means the average return should be $R_{ave} = -3.62\%$ instead of the arithmetic mean 1% that we found earlier.

General definition of geometric mean

Note that in the previous example, if we let $\bar{x} = 1 + R_{ave}, x_1 = 1 + R_1, x_2 = 1 + R_2, \cdots, x_5 = 1 + R_5$, then we would have

$$\bar{x} = \left(\prod_{n=1}^{5} x_i \right)^{\frac{1}{5}}.$$

This is how we define the geometric mean in general.

Geometric mean definition

The geometric mean is defined as the $n-$th root of the product of n numbers, i.e., for a set of numbers $x_1, x_2, ..., x_n$, the geometric mean is defined as

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} = \sqrt[n]{x_1 x_2 \cdots x_n}$$

AM-GM inequality

for any list of n nonnegative real numbers $x_1, x_2, \ldots, x_n$,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}\,,$$

and that equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Proof of AM-GM inequality when $n = 2$

If there are two elements in the series, the AM-GM states this:

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

The proof for this simplified case is quite simple:

Proof

We'll use standard induction method to prove AM-GM inequality.
When $n = 1$: the conclusion holds;
when $n = 2$: we proved the claim above.
We must show that

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}\,.$$

Without loss of generality, we may rescale the x_i so that $x_1 \cdots x_n = 1$. If all $x_i = 1$, the proof is trivial. Thus, assume at least one $x_i > 1$ and one $x_i < 1$; we assume that $x_1 > 1$ and $x_2 < 1$.

By the inductive assumption, we have

$$\frac{x_1x_2 + x_3 + \cdots + x_n}{n-1} \geq \sqrt[n-1]{(x_1x_2)a_3 \cdots x_n} = 1.$$

Thus we have $x_1x_2 + x_3 + \cdots + x_n \geq n - 1$. We need to show

$$x_1 + x_2 + x_3 + \cdots + x_n \geq n.$$

This would follow if $x_1 + x_2 \geq x_1x_2 + 1$. But

$$x_1 + x_2 - (x_1x_2 + 1) = (x_1 - 1)(1 - x_2) \geq 0$$

proving the claim.

$\square$

Application of AM-GM inequality

A farmer wants to fence in 60 000 square meter of land in a rectangular plot along a straight highway. The fence he plans to use along the highway costs \$2 per meter, while the fence for the other three sides costs \$1 per meter. How much of each type of fence will he have to buy in order to keep expenses to a minimum? What is the minimum expense?

Solution